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$$\cot B = \frac{c}{b\sin A} - \cot A\ ;\ a = \frac{b\sin A}{\sin B}.$$

Find logarithm of $b\sin A$ in the column for a; then in the column for B, find $\frac{c}{b\sin A}$ as a logarithm, then as a number; to find B, use the table of natural cotangents.

If a solution wholly logarithmic is desired, take

$$\tan\theta = \frac{b\sin A}{c}\ ;\ \tan B = \frac{\sin A\sin\theta}{\sin(A-\theta)};\ a = \frac{b\sin A}{\sin B}.$$

◆

INTEGRATION OF ELLIPTIC INTEGRALS.

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[Continued from November Number.]

Since all the expressions in the form of

$$\int_0^{2\pi} \sin^a\theta\cos^{2b+1}\theta(1-e^2\sin^2\theta)^c d\theta = 0,$$

(where a, b, c are integers), we will not consider them in our subsequent treatment. From (13_0) we have,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{e^2}[F(e,\ \tfrac{1}{2}\pi) - E(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ldots(3)$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{e^2}[E(e,\ \tfrac{1}{2}\pi) - (1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ldots(4).$$

If $n=0$ in (10_0), $A_2 = \frac{2}{3}[(1+e^2)/e]A_1 - \frac{1}{3}A_0$.

$$\therefore A_2 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos 2\varphi d\varphi}{(1+e^2-2e\cos\theta)^{\frac{1}{2}}}$$

$$=\frac{4}{3\pi e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(14_0).$$

$$\therefore \int_0^{2\pi}\frac{4e^2\sin^4\theta+1-2(1+e^2)\sin^2\theta}{\sqrt{(1-e^2\sin^2\theta)}}d\theta$$

$$=\frac{4}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(15_0).$$

$$\therefore\ 4e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-2(1+e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)].$$

∴ From (1) and (3) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(5).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

∴ From (3) and (5) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2-e^2)E(e,\ \tfrac{1}{2}\pi)-2(1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(6).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

∴ From (4) and (6) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{3e^4}[(2-5e^2+3e^4)F(e,\ \tfrac{1}{2}\pi)-2(1-2e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(7).$$

(15_0) may be written as follows :

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-2\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta-2e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3e^2}[(2+e^2)F(e,\ \tfrac{1}{2}\pi)-2(1+e^2)E(e,\ \tfrac{1}{2}\pi)].$$

$\therefore$ From (1) and (6) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta=\frac{1}{3e^2}[(1-e^2)F(e,\ \tfrac{1}{2}\pi)-(1-2e^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots(8).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}d\theta-\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta.$$

$\therefore$ From (2) and (8) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta=\frac{1}{3e^2}[(1+e^2)E(e,\ \tfrac{1}{2}\pi)-(1-e^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(9).$$

Let $n=1$ in (10_0), then $A_3=\frac{4}{5}.\frac{1+e^2}{e}.A_2-\frac{3}{5}A_1$.

$$\therefore A_3=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos3\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}}$$

$$=\frac{4}{15\pi e^3}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{2}\pi)-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(16_0).$$

From (4_0) we get

$$16e^3\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+12e\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-12e^3\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-3e\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^3}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{2}\pi)$$

$$-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)\ldots\ldots(17_0).$$

$\therefore$ From (3), (6), (7) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{15e^6}[(8+3e^2+4e^4)F(e,\ \tfrac{1}{2}\pi)$$

$$-(8+7e^2+8e^4)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(10).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$\therefore$ From (5) and (10) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{15e^6}[(8-3e^2-2e^4)E(e, \tfrac{1}{2}\pi)-(8-7e^2-e^4)F(e, \tfrac{1}{2}\pi)]. (11)$$

$$\int_0^{2\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \int_0^{2\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

∴ From (6) and (11) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{15e^6}[(8-17e^2+9e^4)F(e, \tfrac{1}{2}\pi)$$

$$-(8-13e^2+3e^4)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (12).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \int_0^{\frac{1}{2}\pi} \frac{\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

∴ From (7) and (12) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{15e^6}[(8-23e^2+23e^4)E(e, \tfrac{1}{2}\pi)$$

$$-(8-27e^2+34e^4-15e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (13).$$

(17_0) may be written as follows :

$$4e^3\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + 12e\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta$$

$$-3e\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{15e^3}[(8+3e^2+4e^4)F(e,\tfrac{1}{2}\pi)-(8+7e^2+8e^4)E(e,\tfrac{1}{2}\pi)].$$

∴ From (10) and (3) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta$$

$$= \frac{1}{15e^4}[2(1-e^2+e^4)E(e, \tfrac{1}{2}\pi)-(2-3e^2+e^4)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (14).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^2\theta d\theta = \int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta$$

$$-\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^4\theta d\theta = \int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^2\theta d\theta$$

$$-\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^4\theta d\theta.$$

From (8), (14) and (9), (14) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\sin^4\theta d\theta = \frac{1}{15e^4}[2(1+e^2-2e^4)F(e, \tfrac{1}{2}\pi)$$

$$-(2+3e^2-8e^4)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (15).$$

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\cos^4\theta d\theta = \frac{1}{15e^4}[2(1-4e^2+3e^4)F(e, \tfrac{1}{2}\pi)$$

$$-(2--7e^2-3e^4)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (16).$$

Let $n=2$ in (10_0), then $A_4 = \frac{6}{7}.\frac{1+e^2}{e}.A_3 - \frac{5}{7}A_2$.

$$\therefore A_4 = \frac{4}{105\pi e^4}[(48+16e^2+17e^4+24e^6)F(e, \tfrac{1}{2}\pi)$$

$$-8(6+5e^2+5e^4+6e^6)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (18_0).$$

∴ By symmetry from A_2, A_3 we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\surd(1-e^2\sin^2\theta)} = \frac{1}{105e^8}[(48+16e^2+17e^4+24e^6)F(e, \tfrac{1}{2}\pi)$$

$$-8(6+5e^2+5e^4+6e^6)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (17).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta d\theta}{\surd(1-e^2\sin^2\theta)} = \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta d\theta}{\surd(1-e^2\sin^2\theta)} - \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\surd(1-e^2\sin^2\theta)}.$$

∴ From (10) and (17) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\surd(1-e^2\sin^2\theta)} = \frac{1}{105e^8}[(48-16e^2-9e^4-8e^6)E(e, \tfrac{1}{2}\pi)$$

$$-4(12-10e^2-e^4-e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (18).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\surd(1-e^2\sin^2\theta)} = \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\surd(1-e^2\sin^2\theta)} - \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\surd(1-e^2\sin^2\theta)}.$$

∴ From (11) and (18) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\surd(1-e^2\sin^2\theta)} = \frac{1}{35e^8}[(16-32e^2+15e^4+e^6)F(e, \tfrac{1}{2}\pi)$$

$$-2(8-12e^2+2e^4+e^6)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (19).$$

[To be Continued.]